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CONSTRUCTION OF LOCAL C^1 QUARTIC SPLINE ELEMENTS FOR OPTIMAL-ORDER APPROXIMATION

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ABSTRACT. This paper is concerned with a study of approximation order and construction of locally supported elements for the space $S_4^1(\Delta)$ of C^1 *pp* (piecewise polynomial) functions on an arbitrary triangulation Δ of a connected polygonal domain Ω in $\mathbb{R}^2$. It is well known that even when Δ is a three-directional mesh $\Delta^{(1)}$, the order of approximation of $S_4^1(\Delta^{(1)})$ is only 4, not 5. The objective of this paper is two-fold: (i) A local Clough-Tocher refinement procedure of an arbitrary triangulation Δ is introduced so as to yield the optimal (fifth) order of approximation, where locality means that only a few isolated triangles need refinement, and (ii) locally supported Hermite elements are constructed to achieve the optimal order of approximation.

1. INTRODUCTION

Let $\Omega \subset \mathbb{R}^2$ be a connected polygonal domain and Δ an arbitrary triangulation of Ω. As usual, $S_k^r(\Delta)$ denotes the subspace of the space $C^r(\Omega)$ of *pp* (:= piecewise polynomial) functions with total degree $\leq k$ over the partition Δ. The *approximation order* of $S_k^r(\Delta)$ is the largest integer ρ for which

$$\operatorname{dist}(f, S_k^r(\Delta)) \leq C|\Delta|^\rho$$

holds for all sufficiently smooth functions f, where the constant C depends only on f and the smallest angle in Δ. Here and throughout, the distance is measured in the supremum norm $\|\cdot\|$ and $|\Delta| := \sup\{\operatorname{diam} \tau : \tau \in \Delta\}$ denotes the meshsize of Δ.

It is well known that for $k \leq 3r+1$ the optimal approximation order of $k+1$ cannot be achieved in general. For instance, de Boor and Jia proved in [2] that if $k \leq 3r+1$ and Δ is the three-direction mesh $\Delta^{(1)}$, the order of approximation of the space $S_k^r(\Delta^{(1)})$ is at most k. In this paper, we introduce a local Clough-Tocher refinement procedure of an arbitrary triangulation Δ in order to achieve the optimal (fifth) order of approximation by C^1 quartic *pp* functions over this locally refined triangulation $\widehat{\Delta}$ of Δ. Here, locality means that the Clough-Tocher triangle is applied only to some isolated triangles in Δ, and as usual, a triangle is called a *Clough-Tocher triangle*, if it is subdivided, by using an interior point (such as the

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centroid of the triangle), into three subtriangles. We will also construct certain locally supported Hermite elements, which will be called star-vertex splines, to achieve this optimal approximation order.

Generation of an optimal mesh is one of the most important facets in finite element modeling. The method of local Clough-Tocher refinement of triangulations introduced in this paper can be undertaken without any element distortion, and our local interpolation schemes will help in drastically decreasing the computational complexity as compared with the standard (global) Clough-Tocher scheme.

For a vertex v in the triangulation Δ, the *degree* of v, denoted by $\deg(v)$, is the number of edges emanating from v. We call a triangulation Δ an *odd-* (*even-*) *degree triangulation* if the degree of any interior vertex in Δ is an odd (even) number. The organization of this paper is as follows. Our local Clough-Tocher refinement algorithm will be introduced in §2. We shall see that the number of local Clough-Tocher refinement steps, if needed, is quite minimal in general. In particular, triangulations Δ such as any odd-degree triangulation and the four-direction mesh $\Delta^{(2)}$ do not even need any refinement in order to achieve the optimal (fifth) order of approximation from $S_4^1(\Delta)$. A refinement of the three-directional mesh $\Delta^{(1)}$ that already admits fifth order of approximation from S_4^1 is shown in Figure 1. In §3, based on this local Clough-Tocher refinement $\widehat{\Delta}$ of Δ, we outline a procedure for constructing a local basis. This local basis will be called a star-vertex spline basis for the space $S_4^1(\widehat{\Delta})$. An explicit scheme of Hermite interpolation from the space $S_4^1(\widehat{\Delta})$ that provides the optimal fifth approximation order will be discussed in §4.

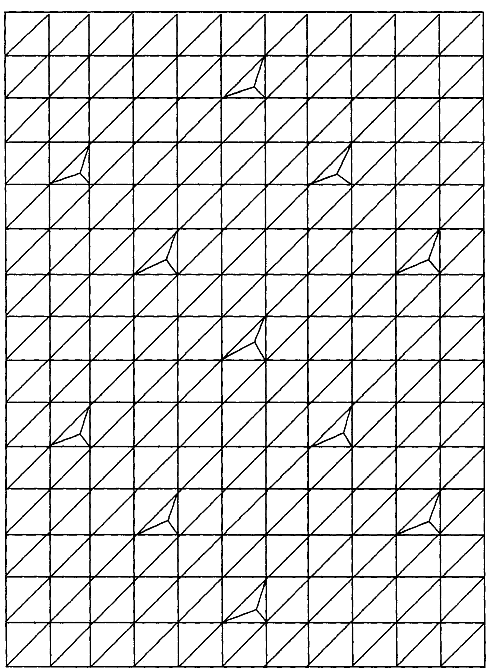

FIGURE 1. A refinement of the three-direction mesh

2. A local Clough-Tocher refinement procedure

For a given triangulation Δ of a polygonal domain $\Omega \subset \mathbb{R}^2$, we need the following notations.

V: the set of all vertices in Δ,
V_I: the set of all interior vertices in Δ,
$V_b := V \setminus V_I$: the set of all boundary vertices in Δ,
E: the collection of all edges in Δ,
E_I: the collection of all interior edges in Δ.

Furthermore, we will use N to denote the total number of triangles in Δ.

We call an interior vertex v a *singular vertex* if (i) its degree is $\deg(v) = 4$ and (ii) v is the intersection of two straight line segments. If e_{j-1}, e_j, e_{j+1} are three consecutive edges with a common vertex v, then the edge e_j is called a *degenerate* edge with respect to v, provided that the two edges e_{j-1} and e_{j+1} are colinear. We consider

V_G: the set of all boundary vertices, all singular vertices, and all interior vertices with odd degrees,

and we call each $v \in V_G$ a *good vertex*. In addition, we will call two vertices in Δ *neighbors* of each other if they are connected by some edge in Δ.

We are now ready to describe an algorithm for constructing a local Clough-Tocher refinement $\widehat{\Delta}$ of an arbitrary triangulation Δ so that the order of approximation from $S_4^1(\widehat{\Delta})$ is full (i.e., five).

Local Clough-Tocher Refinement (LCTR) Algorithm.
Let $V_0 = V \setminus V_G$.
Dowhile $(V_0 \neq \emptyset)$
Pick any vertex v in V_0 and consider its neighbors.
If there exists a neighbor u of v such that $u \in V_G$ or u is a vertex of a Clough-Tocher triangle and that the edge $[u, v]$ is nondegenerate with respect to v,
then delete from V_0 both v and all the other neighbors of u connected to u by nondegenerate edges with respect to themselves.
Call the remaining set the new V_0.
Else, pick any neighbor u of v and subdivide any (but only one) triangle $\tau \in \Delta$ with edge $[u, v]$ into a Clough-Tocher triangle, and delete from V_0 all the vertices of τ as well as all the neighbors of any vertex of τ connected to τ by nondegenerate edges with respect to themselves.
Call the remaining set the new V_0.
Endif
Enddo

The new partition formed by applying the LCTR Algorithm will be denoted by $\widehat{\Delta}$ and called a LCTR of the triangulation Δ. Corresponding to $\widehat{\Delta}$, we use $\widehat{V}$, $\widehat{V}_I$, $\widehat{V}_b$ to denote the set of all vertices, the subset of interior vertices, and the subset of boundary vertices of $\widehat{\Delta}$, respectively. We define $\widehat{E}$, $\widehat{E}_I$, $\widehat{E}_b$ and $\widehat{V}_G$ in a similar way. For any set A, we use the notation $\#A$ for the cardinality of A. A rough upper bound estimate on the number of the refinement steps to form $\widehat{\Delta}$ from Δ is given as follows. From the LCTR Algorithm, it is clear that only a triangle which has either

(i) only nonsingular even-degree interior vertices, or (ii) an edge which is degenerate with respect to a nonsingular interior vertex, may need refinement; and whenever a Clough-Tocher triangle is formed, at least one nonsingular even-degree interior vertex is exempt from further consideration in the LCTR Algorithm. Therefore, the number of refinement steps in the LCTR Algorithm, or equivalently the number of Clough-Tocher triangles added to Δ to form $\widehat{\Delta}$, is bounded from above by

$$L = \min\{\ell, m\},$$

where ℓ is the number of nonsingular even-degree interior vertices in Δ and m is the number of triangles which have either (i) only nonsingular even-degree interior vertices, or (ii) an edge which is degenerate with respect to a nonsingular interior vertex. In particular, if Δ is an odd-degree triangulation (so that $\ell = 0$), or if Δ is a four-direction mesh $\Delta^{(2)}$ (so that $m = 0$), then $\Delta = \widehat{\Delta}$. In other words, for these two types of triangulations Δ, there is no need of refinement at all. A refinement of a three-direction mesh $\Delta^{(1)}$ using the LCTR Algorithm has been shown in Figure 1. Observe that for $\Delta = \Delta^{(1)}$, once a Clough-Tocher triangle is formed by a LCTR, there are generally nine nonsingular even-degree interior vertices that are exempt from further consideration in the LCTR Algorithm.

In general, according to the LCTR Algorithm, we also see that once a Clough-Tocher triangle is added to Δ, at least two nonsingular even-degree interior vertices (in Δ) are changed to odd-degree vertices (in $\widehat{\Delta}$). For any $v \in V_I$, let $\deg_\Delta(v)$ and $\deg_{\widehat{\Delta}}(v)$ denote the degrees of the vertex v in Δ and $\widehat{\Delta}$, respectively. If $\deg_{\widehat{\Delta}}(v) - \deg_\Delta(v) = 0$, then either $v \in V_G$ or v has a neighbor in V_G with odd degree, or else, v is connected to a vertex of a Clough-Tocher triangle. On the other hand, if $\deg_{\widehat{\Delta}}(v) - \deg_\Delta(v) \neq 0$, then v is a vertex of a Clough-Tocher triangle. Observe that a vertex with odd $\deg_\Delta(v)$ might be changed to a vertex with even $\deg_{\widehat{\Delta}}(v)$. In this case, all the neighbors of v are neighboring vertices of a Clough-Tocher triangle. In general, any LCTR $\widehat{\Delta}$ of Δ has the following properties.

Properties of $\widehat{\Delta}$. *Any nonsingular even-degree interior vertex u in $\widehat{\Delta}$ has at least a neighbor of good vertex in $\widehat{V}_G$, or else, u is a neighbor of some vertex of a Clough-Tocher triangle.*

Let σ denote the number of singular vertices in Δ. Then it is well known from [1] that

$$\dim S_4^1(\Delta) = 3\#V_I + 4\#V_b + 3N - \#E_I + \sigma = 3\#V_I + 4\#V_b + \#E + \sigma.$$

On the other hand, since Δ and $\widehat{\Delta}$ have the same number of singular and boundary vertices, we have

$$\dim S_4^1(\widehat{\Delta}) = 3\#\widehat{V}_I + 4\#V_b + \#\widehat{E} + \sigma. \tag{1}$$

In this paper, B-net representations of *pp* functions will play an important role in our discussion. For completeness, we give a very brief review of this topic (more details can be found in [3]). Recall that for any positive integer k, a Bernstein-Bézier polynomial basis of degree k is given by

$$B_{\alpha,T}(x) = \binom{|\alpha|}{\alpha}\xi^\alpha, \quad \alpha = (\alpha_0, \alpha_1, \alpha_2) \in \mathbb{Z}_+^3, \quad |\alpha| := \alpha_0 + \alpha_1 + \alpha_2 = k,$$

where $\xi = (\xi_0, \xi_1, \xi_2)$ is the barycentric ordinate of x with respect to some triangle $\tau = [u, v, w]$ and

$$\xi^\alpha = \xi_0^{\alpha_0}\xi_1^{\alpha_1}\xi_2^{\alpha_2} \quad \text{and} \quad \binom{|\alpha|}{\alpha} = \frac{|\alpha|!}{\alpha_0!\alpha_1!\alpha_2!}.$$

The points

$$x_{\alpha,\tau} = \frac{1}{k}(\alpha_0 u + \alpha_1 v + \alpha_2 w), \qquad |\alpha| = k,$$

are usually called *domain points* of the triangle τ and the set of all domain points on Δ will be denoted by X. For each function $s \in S_k^0(\Delta)$, let

$$s(x) = \sum_{|\alpha|=k} b_{\alpha,\tau} B_{\alpha,\tau}(x), \quad \alpha \in \mathbb{Z}_+^3, \qquad x \in \tau \in \Delta.$$

Then the map

$$b_s \in \mathbb{R}^X : \quad x_{\alpha,\tau} \mapsto b_{\alpha,\tau}, \quad \alpha \in \mathbb{Z}_+^3, \qquad |\alpha| = k, \quad \tau \in \Delta, \tag{2}$$

is called the B-net representation of s. It is well known that to each triangle $\tau \in \Delta$, the matrix

$$(B_{\alpha,\tau}(x_{\beta,\tau}))_{|\alpha|=k,|\beta|=k}$$

is invertible. Thus, the linear system

$$\sum_{|\gamma|=k} c_{\alpha,\gamma} B_{\beta,\tau}(x_{\gamma,\tau}) = \delta_{\alpha,\beta} := \begin{cases} 1, & \alpha = \beta, \\ 0, & \alpha \neq \beta, \end{cases}$$

has a unique solution.

Since this linear system depends only on the barycentric coordinates of $x_{\alpha,\tau}$, the solution $\{c_{\alpha,\beta}\}$ is independent of τ. Let $[\,\cdot\,]$ denote the point-evaluation functional, namely:

$$[x_{\alpha,\tau}]f := f(x_{\alpha,\tau}).$$

Then it is well known (see [4]) that the functionals

$$L_{\alpha,\tau} := \sum_{|\gamma|=k} c_{\alpha,\gamma}[x_{\gamma,\tau}], \quad \alpha \in \mathbb{Z}_+^3, \quad |\alpha| = k,$$

form a dual basis of $\{B_{\alpha,\tau}, |\alpha| = k\}$ in the sense of

$$L_{\alpha,\tau} B_{\beta,\tau} = \delta_{\alpha,\beta}, \quad |\alpha| = |\beta| = k.$$

Furthermore, there is a positive constant C_k, depending only on the degree k, such that

$$\|L_{\alpha,\tau}\| := \sup_{\|f\|_\infty = 1} \|L_{\alpha,\tau} f\|_\infty = \max_{|\beta|=k} |c_{\alpha,\beta}| \leq C_k, \tag{3}$$

for $\alpha \in \mathbb{Z}_+^3$, $|\alpha| = k$.

From (3) and the fact that $b_s(x_{\alpha,\tau}) = L_{\alpha,\tau}s$, we have the following.

Lemma 1. *If $s \in S_k^0(\Delta)$ and $b_s \in \mathbb{R}^X$ is the B-net representation of s, then*

$$\|s\|_\infty \leq \|b_s\|_\infty \leq C_k\|s\|_\infty.$$

Now, let $\tau = [u, v, w]$ and $\tilde{\tau} = [u, v, \tilde{w}]$ be two triangles in Δ with common edge $e = [u, v]$. Also let (c_1, c_2, c_3) denote the barycentric coordinates of $\tilde{w}$ with respect to τ. Then it is well known that the C^1-smoothness conditions across the edge e for $s \in S_4^1(\Delta)$ are determined by the relation

$$b_{\alpha+\mathbf{e}^3,\tilde{\tau}} = c_1 b_{\alpha+\mathbf{e}^1,\tau} + c_2 b_{\alpha+\mathbf{e}^2,\tau} + c_3 b_{\alpha+\mathbf{e}^3,\tau}, \tag{4}$$

where $\alpha = (\alpha_u, \alpha_v, 0) \in \mathbb{Z}_+^3$ with $\alpha_u + \alpha_v = 3$, $\mathbf{e}^1$, $\mathbf{e}^2$, and $\mathbf{e}^3$ denote the standard unit vectors in $\mathbb{R}^3$, and $b_{\alpha,\tau} = b_s(x_{\alpha,\tau})$ is the B-net representation of s as defined in (2).

3. A STAR-VERTEX SPLINE BASIS

A subset $\mathcal{P}$ of domain points will be called a *determining set* of the space $S_k^r(\Delta)$ if and only if every $s \in S_k^r(\Delta)$ is identically zero whenever its B-net representation b_s vanishes on $\mathcal{P}$. Such a determining set $\mathcal{P}$ is called a *minimally determining set* if there is no determining set with fewer elements. Clearly, $\mathcal{P}$ is a determining set for $S_k^r(\Delta)$ if and only if the linear map $s \mapsto b_s|_{\mathcal{P}}$, defined on $S_k^r(\Delta)$, is one-one; also $\mathcal{P}$ is a minimally determining set for $S_k^r(\Delta)$ if and only if this one-one linear map is also onto. To construct a local basis of the space $S_4^1(\widehat{\Delta})$ for a LCTR $\widehat{\Delta}$ of Δ, we choose a minimally determining set $\mathcal{P}$ for $S_4^1(\widehat{\Delta})$ so that the B-net ordinate $b(x)$, $x \in X \setminus \mathcal{P}$, is dependent only on a very small subset of the B-net ordinates that are close to x. This has several important practical advantages: first, the cost of point-evaluation of the interpolant would be less dependent on the amount of data; second, a local change in the data only alters the interpolant locally; and finally, a locally supported basis derived from such a determining set would ensure that the space $S_4^1(\widehat{\Delta})$ has the optimal (fifth) approximation order. To find a determining set for $S_4^1(\widehat{\Delta})$ with these properties, we introduce the following notation.

For any triangle $\tau = [u, v, w] \in \widehat{\Delta}$ with a given vertex $u \in \widehat{V}$, we define, following [1], the set

$$X_{u,\tau}^n = \{x_{\alpha,\tau} : \ \alpha_u = k - n\}$$

of domain points on $\tau \in \widehat{\Delta}$ associated with the vertex u. In addition, for any $u \in \widehat{V}$, we will call

$$R_u^n = \bigcup_{\tau \ni u} X_{u,\tau}^n = \{x_{\alpha,\tau} : \quad \alpha_u = k - n, \quad \tau \in \widehat{\Delta}\}$$

the nth *ring* around u. The corresponding nth *disk* around u is defined by

$$D_u^n = \bigcup_{j=0}^{n} R_u^j = \{x_{\alpha,\tau} : \quad \alpha_u \geq k - n, \quad \tau \in \widehat{\Delta}\}.$$

Next, we introduce the notation of some subsets Y_u^n, $u \in \widehat{V}$, as follows.

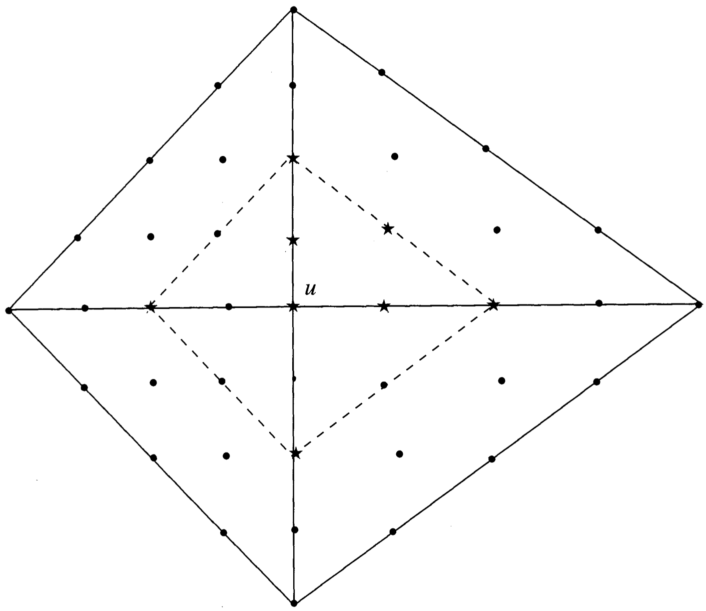

FIGURE 2. The points in Y_u^n, $n = 0, 1, 2$, where u is a singular vertex

(A) Let $n = 0, 1$. For each $u \in \widehat{V}$, we choose a triangle $\tau = [u, v, w]$ attached to u and define

$$Y_u^n := X_{u,\tau}^n. \tag{5}$$

(B) Let $n = 2$.

(i) If $u \in \widehat{V}_b$ or if u is a singular vertex (see Figure 2), then we define

$$Y_u^2 := X_{u,\tau}^2 \cup \left(R_u^2 \cap (\bigcup_{e \in E_u} e) \right), \tag{6}$$

where E_u denotes the collection of all edges with common vertex $u \in V$.

(ii) Let u be a nonsingular even-degree vertex in $\widehat{V}_I$ (see Figure 3 on next page). According to the LCTR Algorithm, if $u \in \widehat{V} \setminus \widehat{V}_G$, then we can choose an edge $e_C = [u, v] \in E_u$, which is nondegenerate with respect to u and is not an edge of τ as already selected in (5) (where τ in (5) is adjusted if necessary) such that either $v \in \widehat{V}_G$ or else, v is a vertex of a Clough-Tocher triangle. If there is a Clough-Tocher triangle τ_u attached to u, then we always select $e_C = [u, \widehat{u}]$, where $\widehat{u}$ is the interior vertex in the Clough-Tocher triangle τ_u. Let E_C denote the collection of all such edges e_C. Then we may define

$$Y_u^2 := X_{u,\tau}^2 \cup \left(R_u^2 \cap (\bigcup_{e \neq e_C, e \in E_u} e) \right). \tag{7}$$

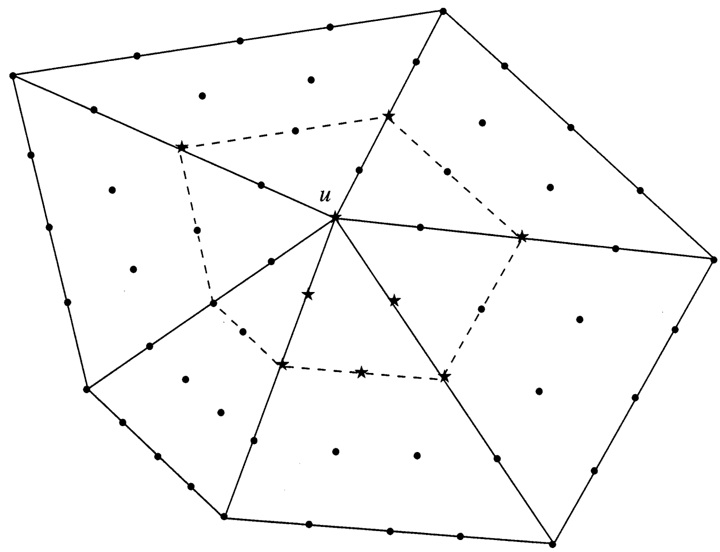

FIGURE 3. The points in Y_u^n, $n = 0, 1, 2$, where $u \notin \widehat{V}_G$

(iii) If $u \in \widehat{V}_I$ is an odd-degree vertex (see Figure 4), then we define

$$Y_u^2 := R_u^2 \cap \Big(\bigcup_{e \in E_u} e \Big). \tag{8}$$

Finally, we set

$$\mathcal{P}_u^2 := \bigcup_{n=0}^{2} Y_u^n, \quad u \in \widehat{V}. \tag{9}$$

Let $x_{_C}$ denote the center of the edge $e_{_C}$ and define

$$\mathcal{P} := \Big(\bigcup_{u \in \widehat{V}} \mathcal{P}_u^2 \Big) \setminus \Big(\bigcup_{e_{_C} \in E_{_C}} x_{_C} \Big). \tag{10}$$

Then we will see that $\mathcal{P}$ is a minimally determining set for the space $S_4^1(\widehat{\Delta})$, as follows.

Theorem 1. *For each $b : \mathcal{P} \mapsto \mathbb{R}$, there exists a unique $g \in S_4^1(\widehat{\Delta})$ such that the B-net representation b_g of g satisfies*

$$b_g|_{\mathcal{P}} = b.$$

To prove Theorem 1, we need the following lemma.

Lemma 2. *For any $u \in \widehat{V}$, the set $\mathcal{P}_u^2$ defined in* (9) *uniquely determines those functions in $S_4^1(\widehat{\Delta})$ that have identical B-net ordinates on D_u^2.*

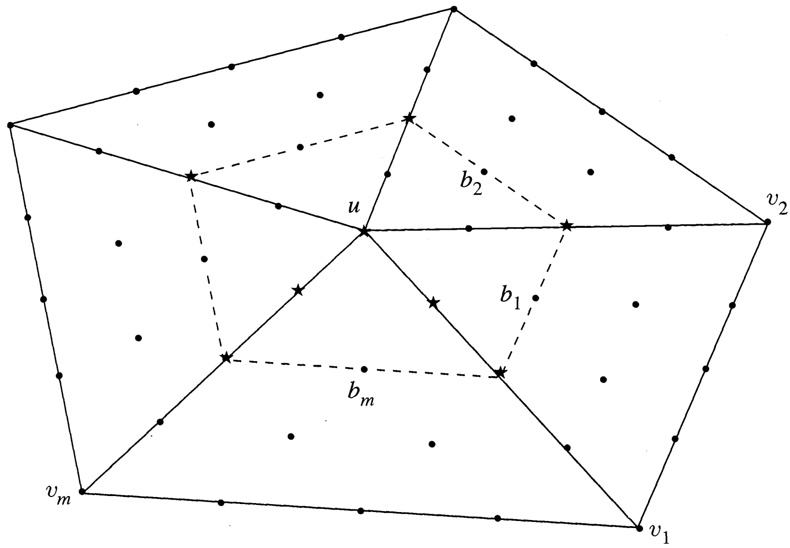

FIGURE 4. The points in Y_u^n, $n = 0, 1, 2$, with odd values of $\deg(u)$

Proof. The proof of this lemma depends on Lemmas 2 – 4 and 6 in [1]. In fact, it suffices to show that b_s vanishes on D_u^2 whenever it vanishes on $\mathcal{P}_u^2$.

For a boundary vertex $u \in \widehat{V}_b = V_b$, this follows by the smoothness condition directly .

Now, suppose that u is either a singular vertex (cf. Figure 2) or a nonsingular even-degree interior vertex (cf. Figure 3). Since $\mathcal{P}_u^2$ contains three noncolinear points in D_u^1, b_s must be zero on D_u^1 according to the C^1-smoothness condition. It is easy to see that by the smoothness condition and the fact that e_C is non-degenerate with respect to u, the remaining B-net ordinates in R_u^2 are also zero.

For an odd-degree vertex u (cf. Figure 4), it follows by the smoothness condition that the zero b_s-values on $\mathcal{P}_u^2$ force all of the b_s-values on D_u^1 to be zero. By writing out explicitly the coefficients in terms of the ratios of (signed) areas in the smoothness condition (4), it is easy to verify that the determinant of the coefficient matrix for the remaining unknowns is 2. Therefore, all the other B-net ordinates on D_u^2 must also be zero. This completes the proof of the lemma. $\square$

Proof of Theorem 1. It is easy to see that there are $(3 + \deg(u))$ points in $\mathcal{P}_u^2$ for a nonsingular interior vertex u, and $(4 + \deg(v))$ points in $\mathcal{P}_v^2$ for a singular or boundary vertex v. Furthermore, it follows from (1) that

$$\#\mathcal{P} = 3\#\widehat{V}_0 + 4\#V_b + \#\widehat{E} + \sigma = \dim S_4^1(\widehat{\Delta}).$$

Thus, if we can prove that $\mathcal{P}$ is a determining set for $S_4^1(\widehat{\Delta})$, then $\mathcal{P}$ is also a minimally determining set of $S_4^1(\widehat{\Delta})$. For this purpose, let us arrange the vertices in $\widehat{V}$ in an appropriate order, and extend the B-net ordinates b_g from b as follows:

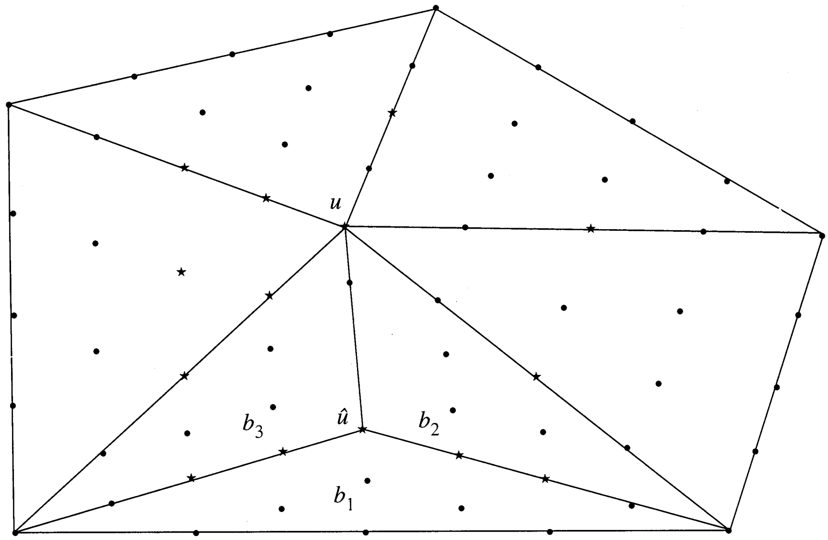

FIGURE 5. The determining set (points $\star$) of $D_u^2 \cup D_{\hat{u}}^2$

(i) For every nonsingular even-degree interior vertex u, which is not a vertex of any Clough-Tocher triangle, according to our choice of e_C in (7), the edge e_C is nondegenerate with respect to u. By Lemma 2, we can determine the b_g-values on all the domain points in D_u^2 from the given values on $\mathcal{P}$.

(ii) Each remaining nonsingular even-degree interior vertex u is also a vertex of some Clough-Tocher triangle τ_u. According to our choice of e_C in (7), the edge e_C is an interior edge of τ_u and so it is nondegenerate with respect to u. Note that all the b_g-values on $\bigcup_{e \in E_u} R_u^2 \cap e$ are either given, or else, are determined in (i). Thus, by Lemma 2, the b_g-values on all the domain points in D_u^2 are determined.

(iii) The remaining vertices are now in $\widehat{V}_G$, which contains all the vertices in $\widehat{V} \setminus V$.

From (i), (ii), and the choice of Y_u^2, we see that all the middle points of the edges have been uniquely determined. Figure 5 illustrates the case of the centroid $\widehat{u} \in \widehat{V}_G$ of a Clough-Tocher triangle, which is connected to an even-degree vertex u. Therefore, by Lemma 2, it is clear that the b_g-values are uniquely determined on all the domain points in D_u^2.

We see that b_g satisfies a C^1-smoothness condition on D_u^2. Since $(\#\mathcal{P}) = \dim S_4^1(\widehat{\Delta})$, it is also clear that such an extension is unique. This completes the proof of the theorem. $\square$

Theorem 1 implies that $\mathcal{P}$ is a minimally determining set of $S_4^1(\widehat{\Delta})$. Let

$$d := \dim S_4^1(\widehat{\Delta}),$$

and write

$$\mathcal{P} = \{x_1, \cdots, x_d\} \subset X.$$

Also, let $\{b_1, \ldots, b_d\} \subset \mathbb{R}^X$ be the "dual" of $\mathcal{P}$, defined by the following: (i) $b_i(x_j) = \delta_{ij}, \quad i,j = 1, \ldots, d$, and (ii) for each $x \in X \setminus \mathcal{P}$, $b_i(x)$ is uniquely determined by the smoothness condition (4) and the procedure described in the proof of Theorem 1. Let $s_i \in S_4^1(\widehat{\Delta})$, with B-net representation b_i, $i = 1, \ldots, d$. Then $\{s_1, \ldots, s_d\}$ is a basis of $S_4^1(\widehat{\Delta})$.

We denote by $\overline{\mathrm{St}}(u)$ the closed star of the vertex u in a triangulation Δ [5, p.135]; i.e., the cell formed by all the triangles in Δ with u as the common vertex, and call it the 1-star $\overline{\mathrm{St}}^1(u)$ of u. For $m \geq 1$, the m-star $\overline{\mathrm{St}}^m(u)$ of u is then defined to be the union of all the triangles in Δ which have at least one common vertex with the $(m-1)$-star $\overline{\mathrm{St}}^{m-1}(u)$. Similar to the definition of vertex splines, a spline is called a *m-star vertex spline* if its support is no larger than $\overline{\mathrm{St}}^m(u)$ for some vertex $u \in \Delta$. We have the following result.

Theorem 2. *The basis $\{s_1, \cdots, s_d\}$ of $S_4^1(\widehat{\Delta})$ defined as above is a locally supported basis. Furthermore, for each $i = 1, \ldots, d$, there is some $u_i \in V$ such that*

$$\mathrm{supp}(s_i) \subset \overline{\mathrm{St}}^3(u_i).$$

Proof. Following the procedure described in the proof of Theorem 1, we can see that the b_i-values of s_i are uniquely determined on X. We divide our discussion into three cases.

(i) For $x_i \in \mathcal{P} \cap D_u^1$, it is clear that $\mathrm{supp}(s_i) \subset \overline{\mathrm{St}}(u)$, since $b_i = 0$ outside of $\overline{\mathrm{St}}(u)$.

Now we assume $x_i \in \mathcal{P} \cap R_u^2$.

(ii) Suppose $x_i \in D_u^2$ and $u \in \widehat{V}_G$. If x_i is not the midpoint of an edge, then $\mathrm{supp}(s_i) \subset \overline{\mathrm{St}}(u)$. On the other hand, if x_i is the midpoint of some edge $[u, v]$ and v is an even-degree nonsingular interior vertex, and if an edge $[v, w]$ is chosen to be e_C as in (7) for the vertex v, then

$$\mathrm{supp}(s_i) \subset \overline{\mathrm{St}}(u) \cup \overline{\mathrm{St}}(v) \cup \overline{\mathrm{St}}(w) \subset \overline{\mathrm{St}}^2(v).$$

Otherwise, we have $v \in \widehat{V}_G$ and

$$\mathrm{supp}(s_i) \subset \overline{\mathrm{St}}(u) \cup \overline{\mathrm{St}}(v) \subset \overline{\mathrm{St}}^2(u).$$

(iii) Now, suppose u is an even-degree nonsingular interior vertex. If x_i is not the midpoint of an edge in $\widehat{E}$, then $\mathrm{supp}(s_i) \subset \overline{\mathrm{St}}(u)$. If x_i is the midpoint of some edge $[u, u']$ where $u' \in \widehat{V}_G$, then similar to (ii), there is an edge $[u, v]$ chosen to be e_C as in (7) for the vertex u, and

$$\mathrm{supp}(s_i) \subset \overline{\mathrm{St}}(u) \cup \overline{\mathrm{St}}(v) \cup \overline{\mathrm{St}}(u') \subset \overline{\mathrm{St}}^2(u).$$

Otherwise, by the choice of the determining set in $\mathcal{P}$, there are edges $[u, v]$ and $[u', v']$ defined to be e_C as in (7) for the vertices u and u', respectively, such that v and v' are vertices in $\widehat{V}_G$ and

$$\mathrm{supp}(s_i) \subset \overline{\mathrm{St}}(u) \cup \overline{\mathrm{St}}(u') \cup \overline{\mathrm{St}}(v) \cup \overline{\mathrm{St}}(v') \subset \overline{\mathrm{St}}^3(u).$$

In summary, for any $x_i \in \mathcal{P}$, its corresponding $s_i \in S_4^1(\widehat{\Delta})$ has support $\mathrm{supp}(s_i) \subset \overline{\mathrm{St}}^3(u)$ for some vertex $u \in V$. This completes the proof of the theorem. $\square$

From the proof of Theorem 2, we can actually see that $S_4^1(\widehat{\Delta})$ has basis functions whose support is no larger than

$$\mathrm{supp}(s_i) \subset \overline{\mathrm{St}}(u) \cup \overline{\mathrm{St}}(u') \cup \overline{\mathrm{St}}(v) \cup \overline{\mathrm{St}}(v')$$

for four consecutive vertices v, u, u' and v'.

4. Interpolation scheme and its approximation power

In this section, we construct an explicit interpolation scheme to prove that the space $S_4^1(\widehat{\Delta})$ achieves its optimal approximation order. Since the minimally determining set $\mathcal{P}$ contains the domain points in $X_{u,\tau}^n$, $n = 0, 1$ for each $u \in \widehat{V}$ and some triangle τ attached to u, the interpolation scheme can be chosen to interpolate the function values as well as gradient values of a given $f \in C^1(\Omega)$ at each sample point, as follows.

Interpolation Scheme.

Step 1. For each vertex $u \in \widehat{V}$, let $\tau = [u, v, w]$ be the corresponding triangle associated with Y_u^n, $n = 0, 1$, and p_u the Hermite polynomial that interpolates f at the vertex u on τ; that is,

$$\begin{cases} p_u(u) = f(u), \\ D_i p_u(u) = D_i f(u), \qquad i = 1, 2, \end{cases}$$

where D_1 and D_2 denote the directional derivatives along the directions $e_1 = v - u$ and $e_2 = w - u$. Consider the B-net representation

$$p_u = \sum_{|\alpha|=k} b_{p_u}(x_{\alpha,\tau}) B_{\alpha,\tau},$$

and set

$$b_g(x) = b_{p_u}(x), \qquad x \in Y_u^n, \quad n = 0, 1.$$

Step 2. Choosing $b_g(x) = b_{p_u}(x)$, $x \in Y_u^2$, in the order as described in the proof of Theorem 1, we determine the remaining b_g-values on $X \setminus \mathcal{P}$ by applying the smoothness condition (4).

Denote by T the linear operator obtained by the *Interpolation Scheme*:

$$T: \ f \mapsto g, \ f \in C^1(\widehat{\Delta}). \tag{11}$$

It is clear from the construction and the choice of the determining set $\mathcal{P}$ that T is well defined.

Let a denote the smallest angle among all the triangles in $\widehat{\Delta}$, and let C_a denote a constant depending only on a, which may be different from situation to situation. For a triangle $\tau \in \widehat{\Delta}$ with vertex u, v and w, we define a neighborhood of τ as

$$\Omega(\tau) = \overline{\mathrm{St}}^2(u) \cup \overline{\mathrm{St}}^2(v) \cup \overline{\mathrm{St}}^2(w). \tag{12}$$

Then we have the following.

Lemma 3. *The linear operator T defined in* (11) *satisfies*
(i) $Tp = p$ *for any polynomial* $p \in \pi_4$, *and*
(ii) $\|Tf|_\tau\|_\infty \le C_a \|f|_{\Omega(\tau)}\|_\infty$.

Proof. The first part of the lemma is obvious by the construction of the operator T. The supports of the basis functions $\{s_i\}_{i=1}^d$ of $S_4^1(\widehat{\Delta})$ satisfy

$$\operatorname{supp}(s_i) \subset \Omega(\tau) \quad \text{for some} \quad \tau \in \widehat{\Delta},$$

from the proof of Theorem 2. Let $g(x) = Tf(x) := \sum_i c_i s_i(x)$, $x \in \tau$, $\tau \in \widehat{\Delta}$. According to Theorem 2, we have $s_i(x) \neq 0$ only if the corresponding domain point x_i lies in $\Omega(\tau)$. Therefore, the number of nonzero values of the c_i's is bounded from above by C_a. Moreover, by Lemma 1 and Theorem 2, we have $\|s_i\| \le C_a \max_{y\in\Omega(\tau)\cap\mathcal{P}} |b_{s_i}(y)| = C_a$. From the definition of s_i, we also have

$$c_i = b_g(x_i), \qquad x_i \in \mathcal{P}.$$

Thus, it follows from Lemma 1 that

$$|Tf(x)| \le C_a \max_{x\in\Omega(\tau)\cap\mathcal{P}} |b_g(x)| \le C_a \|g(x)|_{\Omega(\tau)}\| \le C_a \|f|_{\Omega(\tau)}\|_\infty, \quad x \in \tau \in \widehat{\Delta}.$$

The last inequality holds because $g|_\tau$ is a Hermite interpolation polynomial on each triangle $\tau \in \widehat{\Delta}$, and that from the B-net representation the operator (on τ) so defined is bounded by a constant independent of the shape of τ. This completes the proof of the lemma. □

We are now in a position to prove the following main result of this paper.

Theorem 3. *The linear operator T defined in* (11) *has the optimal* (*fifth*) *order of approximation; that is,*

$$\|Tf - f\| \le C_a \|f^{(5)}\| \, |\widehat{\Delta}|^5, \quad f \in C^5(\widehat{\Delta}).$$

Consequently,

$$\operatorname{dist}(f, S_4^1(\widehat{\Delta})) \le C_a \|f^{(5)}\| \, |\widehat{\Delta}|^5, \quad f \in C^5(\widehat{\Delta}),$$

where $|\widehat{\Delta}|$ is the meshsize of $\widehat{\Delta}$.

Proof. Fix any $\tau \in \widehat{\Delta}$ and any $x \in \tau$. Let $f \in C^5(\widehat{\Delta})$ and consider a polynomial $p \in \pi_4$ that interpolates f at point x, namely,

$$p(x) = f(x), \tag{13}$$

and

$$|f(y) - p(y)| \le C \|f^{(5)}\| \, |\widehat{\Delta}|^5, \qquad y \in \Omega(\tau), \tag{14}$$

where C is an absolute constant. By appying (13), Lemma 3, and (14) consecutively, it follows that

$$|f(x) - Tf(x)| = |T(f-p)(x)| \le C_a \|(f-p)|_{\Omega(\tau)}\| \le C_a \|f^{(5)}\| \, |\widehat{\Delta}|^5.$$

Since this inequality holds for any $x \in \widehat{\Delta}$, we have

$$\|Tf - f\| \leq C_a \|f^{(5)}\| \, |\widehat{\Delta}|^5.$$

This completes the proof of the theorem. □

If the original triangulation Δ satisfies the condition that for each vertex $v \in V$, $\deg(v)$ is an odd number, or v is a singular vertex, then we see from the LCTR Algorithm that $\widehat{\Delta} = \Delta$. Also, we have $\widehat{\Delta} = \Delta$ for the four-direction mesh $\Delta^{(2)}$. In both cases, we can choose the minimally determining set $\mathcal{P}$ to contain midpoints of all the edges in E.

Corollary 1. (a) *If a triangulation Δ contains only odd-degree interior vertices or singular vertices, then there is a Hermite interpolation scheme to achieve the optimal approximation order of the space $S_4^1(\Delta)$.*

(b) *If Δ is a four-direction mesh $\Delta^{(2)}$, then the space $S_4^1(\Delta^{(2)})$ has fifth order of approximation, and there is a Hermite interpolation scheme that achieves this optimal approximation order.*

Bibliography

1. P. Alfeld, B. Piper, and L. L. Schumaker, *An explicit basis for C^1 quartic bivariate splines*, SIAM J. Numer. Anal. **24** (1987), 891–911. MR **88i:**41014
2. C. de Boor and R. Q. Jia, *A sharp upper bound on the approximation order of smooth bivariate pp functions*, J. Approx. Theory **72** (1993), 24–33. MR **94e:**41012
3. C. K. Chui, *Multivariate splines*, CBMS Series in Applied Mathematics, vol. 54, SIAM, Philadelphia, PA, 1988. MR **92e:**41009
4. Z.R. Guo and R.Q. Jia, *A B-net approach to study of multivariate splines*, Adv. Math. **19** (1990), 189–198. MR **91c:**41024
5. J.J. Rotman, *An introduction to algebraic topology*, Graduate Texts in Math., vol. 119, Springer-Verlag, New York, 1988. MR **90e:**55001

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